UNIVERSITY OF WATERLOO
Faculty of Mathematics

VORTEX SOLUTIONS ON RIEMANN SURFACES FROM HYPERBOLIC TESSELLATIONS

Department of Pure Mathematics
University of Waterloo
Waterloo, Ontario

Prepared by
Caleb Suan
3A Pure Mathematics
ID 20506127
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MEMORANDUM

To: Benoit Charbonneau

From: Caleb Suan

Date: August 14, 2018

Re: Work Report: Vortex Solutions on Riemann Surfaces from Hyperbolic Tessellations

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As we agreed, I have prepared the enclosed report, "Vortex Solutions on Riemann Surfaces from Hyperbolic Tessellations," for my 3A work report and for the Department of Pure Mathematics. This report, the last of four work reports that the Co-operative Education Program requires that I successfully complete as part of BMath Co-op degree requirements, has not received academic credit.

The research project that we undertook was finding symmetric vortex solutions on Riemann surfaces. My role as research assistant required that I search for relevant papers and articles on the topic and explore possible methods and ansätzes to find such solutions. This report is a study of a method which Maldonado and Manton used to find analytic abelian Higgs vortex solutions on compact Riemann surfaces.

The Faculty of Mathematics requests that you evaluate this report for command of topic and technical content and analysis. Following your assessment, the report, together with your evaluation, will be submitted to the Math Undergraduate Office for evaluation on campus by qualified work report markers. The combined marks determine whether the report will receive credit and whether it will be considered for an award.

Thank you for your assistance in preparing this report.

Caleb Suan

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Executive Summary

The study of vortices in Ginzburg-Landau theory has been relatively popular in the past few decades. Both mathematicians and physicists have looked at these objects from their respective perspectives. In particular, the work of Baptista, Maldonado, and Manton in recent years has shown new ways to think about vortices as well as new ways to create vortex solutions in certain settings.

This paper looks at a specific method used by Maldonado and Manton to create symmetric vortex solutions on compact Riemann surfaces. Though their methodology is outlined in their paper, much of the detail in the construction and well-definition of the solution is glossed over. This paper briefly introduces the setting of the abelian Higgs model and vortices before using complex analytic methods to fill in the gaps in Maldonado and Manton's paper. In particular, it motivates the well-definition of the construction. Additionally, this paper also briefly touches on how this method may be applied to generalizations of the model and on how other vortex solutions may be found.

1.0 Introduction

Vortices correspond to states which minimize the Yang-Mills-Higgs energy functional and may be defined on a variety of surfaces. The study of such objects has stemmed from Ginzburg-Landau theory from physics, which describes superconductivity and research in this area has been quite active in recent decades both from a mathematics and physics viewpoint.

This paper focuses on vortices in the abelian Higgs model on Riemann surfaces, which consist of a $U(1)$ connection A on a line bundle as well as a section or Higgs field ϕ, and aim to describe methods in which solutions may be found. In particular, a method of constructing vortex solutions on compact Riemann surfaces using a method of Maldonado and Manton [7] will be studied. Generalizations of this model suggested by Manton in [8] will also be discussed.

2.0 Abelian Higgs Vortices

2.1 The Abelian Higgs Model

Let Σ be a Riemann surface, which is a connected 1-complex-dimensional manifold, with a metric compatible with the complex structure. Locally, the complex coordinates $z = x+iy$ may be used and the metric may be written as

$$ds^2 = \Omega dz \wedge d\bar{z} \tag{1}$$

where Ω is a conformal factor. Additionally, suppose L is a $U(1)$ line bundle on Σ.

Given a $U(1)$ connection A on L, it may be represented in coordinates by a real 1-form $A = A_z dz + A_{\bar{z}} d\bar{z}$. A has curvature $F = dA + A \wedge A$, however since the gauge group $U(1)$ is abelian, the $A \wedge A$ term vanishes, giving

$$F = F_{z\bar{z}} dz \wedge d\bar{z} = (\partial_z A_{\bar{z}} - \partial_{\bar{z}} A_z) dz \wedge d\bar{z}. \tag{2}$$

Moreover, a section ϕ on L may locally be considered as a complex-valued function. Lastly, suppose that the first Chern number of the bundle,

$$N = \frac{1}{2\pi} \int_\Sigma F \tag{3}$$

is a positive integer.

The Yang-Mills-Higgs energy functional in the abelian Higgs model is given by

$$E(A, \phi) = \int_\Sigma \left[(\star F)^2 + \frac{1}{\Omega} |D\phi|^2 + (1 - |\phi|^2)^2 \right] \frac{i}{2} \Omega dz \wedge d\bar{z} \tag{4}$$

where $\star$ is the Hodge star operator on Σ, and D is the gauge covariant derivative with respect to the connection A. Alternatively, in coordinates, the functional reads

$$E(A, \phi) = \int_\Sigma \left[\left(\frac{-2iF_{z\bar{z}}}{\Omega} \right)^2 + \frac{1}{\Omega} (|D_z \phi|^2 + |D_{\bar{z}} \phi|^2) + (1 - |\phi|^2)^2 \right] \frac{i}{2} \Omega dz \wedge d\bar{z} \tag{5}$$

Completing the square using the first and last terms in the integral yields

$$\begin{aligned} E(A, \phi) = &\int_\Sigma \left[\left(\frac{-2iF_{z\bar{z}}}{\Omega} - (1 - |\phi|^2) \right)^2 + \frac{1}{\Omega} (|D_z \phi|^2 + |D_{\bar{z}} \phi|^2) \right] \frac{i}{2} \Omega dz \wedge d\bar{z} \\ &+ \int_\Sigma \left[-4iF_{z\bar{z}} + 4iF_{z\bar{z}} |\phi|^2 \right] \frac{i}{2} dz \wedge d\bar{z} \end{aligned} \tag{6}$$

Reducing the second term in equation (5) using a completing-the-square-like method then gives

$$
\begin{aligned}
E(A,\phi) = & \int_\Sigma \left[\left(\frac{-2iF_{z\overline{z}}}{\Omega} - (1-|\phi|^2) \right)^2 + \frac{1}{\Omega}|D_{\overline{z}}\phi|^2 \right] \frac{i}{2}\Omega dz \wedge d\overline{z} \\
& + \int_\Sigma \left[-4iF_{z\overline{z}} + 4iF_{z\overline{z}}|\phi|^2 - 4iF_{z\overline{z}}|\phi|^2 - 2id(\overline{\phi}D\phi) \right] \frac{i}{2} dz \wedge d\overline{z}
\end{aligned} \tag{7}
$$

Using Stokes' Theorem, the total derivative term $d(\overline{\phi}D\phi)$ vanishes. This combined with (3) shows that the second integral may be simplified

$$
\int_\Sigma (-4iF_{z\overline{z}}) \frac{i}{2} dz \wedge d\overline{z} = \int_\Sigma F = 2\pi N \tag{8}
$$

Hence the Yang-Mills-Higgs energy functional may be written as

$$
E(A,\phi) = \int_\Sigma \left[\left(\frac{-2i}{\Omega} F_{z\overline{z}} - (1-|\phi|^2) \right)^2 + \frac{2}{\Omega}|D_{\overline{z}}\phi|^2 \right] \frac{i}{2}\Omega dz \wedge d\overline{z} + 4\pi N. \tag{9}
$$

Since the integrand is the sum of non-negative terms, it is clear that the function attains its minimum only when A and ϕ satisfy

$$
-\frac{2i}{\Omega} F_{z\overline{z}} = 1 - |\phi|^2 \tag{10}
$$

$$
D_{\overline{z}}\phi = 0 \tag{11}
$$

The first-order equations (10) and (11) obtained are called the Bogomolny vortex equations [2]. Equation (10) may also be written more invariantly as

$$
\star F = 1 - |\phi|^2. \tag{12}
$$

2.2 The Vortex Equations and Taubes Equation

The Bogomolny equations (10) and (11) may be uncoupled by first looking at the latter equation. Expanding out the covariant derivative term gives $\partial_{\bar{z}}\phi - iA_{\bar{z}}\phi = 0$. Rearranging it gives a formula for $A_{\bar{z}}$.

$$A_{\bar{z}} = -i\frac{\partial_{\bar{z}}\phi}{\phi} = -i\partial_{\bar{z}} \log \phi \tag{13}$$

Since A is a $U(1)$ connection, $\overline{A_z} = A_{\bar{z}}$. Hence $A_z = i\partial_z \log \overline{\phi}$. Applying equation (2) yields a formula for $F_{z\bar{z}}$ given by

$$\begin{aligned} F_{z\bar{z}} &= \partial_z A_{\bar{z}} - \partial_{\bar{z}} A_z \\ &= \partial_z(-i\partial_{\bar{z}} \log \phi) - \partial_{\bar{z}} i\partial_z \log \overline{\phi} \\ &= -i\partial_z\partial_{\bar{z}} \log |\phi|^2 \\ &= -i\frac{1}{4}\nabla^2 \log |\phi|^2 \end{aligned} \tag{14}$$

where $\nabla^2 = \partial_x\partial_y$ is the standard Euclidean Laplacian. Substituting (14) back into the first Bogomolny equation (10) results in Taubes' equation [8] [9]

$$-\frac{1}{2\Omega}\nabla^2 \log |\phi|^2 = 1 - |\phi|^2 \tag{15}$$

Taubes' equation allows one to solve directly for ϕ up to a choice of phase. This, in turn, gives a solution for A, which may be determined using the Bogmolny equations (10) (11).

For simplicity, make the substitution $|\phi|^2 = e^{2u}$ with u being a real function. In order to do this, it must first be noted that Taubes' equation (15) is only valid where ϕ is non-zero and extending the domain of Taubes' equation onto the zeros of ϕ requires supplementary delta terms need to be added. After substituting, the Taubes' equation reads

$$-\frac{1}{\Omega}\nabla^2 u = 1 - e^{2u} \tag{16}$$

2.3 Degenerate Metrics

As noted by Baptista in [1], it is worth considering the degenerate metric

$$ds'^2 := |\phi|^2 \Omega dz d\bar{z} = e^{2u}\Omega dz d\bar{z} \tag{17}$$

which will henceforth be referred to as the Baptista metric. The Baptista is not a metric in the traditional sense, as it is degenerate at the zeros of ϕ. From the second Bogomolny equation, we see that ϕ must be a holomorphic section with respect to the complex structure on L induced by A. From this, it must that the zeros of ϕ, and hence those of $|\phi|^2$, must occur at isolated points of Σ. It can also be shown that ϕ has N zeros counted with multiplicity [8].

Looking at the Gaussian curvatures of Σ with metric ds^2 that of Σ with the Baptista metric ds'^2 results in

$$K = -\frac{1}{2\Omega}\nabla^2 \log\Omega \tag{18}$$

$$K' = -\frac{1}{2e^{2u}\Omega}\nabla^2(2u + \log\Omega) \tag{19}$$

where K and K' are the curvatures with metrics ds^2 and ds'^2 respectively. Combining and rearranging gives

$$-K + K'e^{2u} = -\frac{1}{\Omega}\nabla^2 u \tag{20}$$

By comparing equation (20) with Taubes' equation (16) it can be seen that

$$1 - e^{2u} = -K + K' e^{2u} \tag{21}$$

2.4 Integrable Cases

The case where the metric ds^2 has constant Gaussian curvature $K = -1$ has been shown to be integrable [8]. In this case, equation (21) says that the curvature of Σ with the Baptista metric must also have constant value $K' = -1$. These constraints allow for the curvature equation (19) to reduce to Liouville's equation as shown below.

First, make another change of variable by asserting that the Baptista metric ds'^2 has conformal factor $e^{2v} := \Omega' = e^{2u}\Omega$. Substituting this back into equation (19) gives

$$K' = -\frac{1}{2e^{2v}} \nabla^2 2v \tag{22}$$

which may be rearranged to get Liouville's equation

$$\nabla^2 v = -e^{2v} \tag{23}$$

On a simply connected domain, Liouville's equation (23) has solutions given by

$$e^{2v} = \Omega' = \frac{4}{(1 - |h(z)|^2)^2} \left| \frac{dh}{dz} \right|^2 \tag{24}$$

where h is an arbitrary local holomorphic function [5].

Locally, there is an expression for Ω is given by

$$\Omega = \frac{4}{(1-|z|^2)^2} \tag{25}$$

Since $|\phi|^2 = e^{2u}$ was the ratio of the two metrics, dividing equation (24) by equation (25) yields a coordinate expression for $|\phi|^2$

$$|\phi|^2 = \frac{\Omega'}{\Omega} = \frac{(1-|z|^2)^2}{(1-|h(z)|^2)^2}\left|\frac{dh}{dz}\right|^2 \tag{26}$$

Thus finding a suitable holomorphic function h on Σ will result in a vortex solution up to a choice of gauge. Additionally, if one were to prescribe points $p_1, ..., p_n$ at which ϕ were to vanish, the function h would need to have ramification points at the p_i.

For example, in [10], Witten constructed vortices on the hyperbolic plane with finite Blaschke products. Using the Poincaré disk model of the hyperbolic plane, the holomorphic function h was chosen to be of the form

$$h(z) = \prod_{m=1}^{N+1} \frac{z - a_m}{1 - \overline{a_m} z} \tag{27}$$

where each a_m satisfies $|a_m| < 1$.

2.5 Vortices from Hyperbolic Tessellations

The local formula from equation (26) was used by Maldonado and Manton to describe a method of creating analytic hyperbolic vortex solutions satisfying certain symmetry properties [7]. In their paper, they considered regular tessellations of the hyperbolic disk. This was due to the result of the Uniformization Theorem stated below.

Theorem 2.0.1 (Uniformization Theorem)**.** *Every Riemann surface is the quotient of a free, proper, and holomorphic action of a discrete group on its universal covering. This universal covering is biholomorphic to one of:*

1. the Riemann sphere, S^2

2. the complex plane, $\mathbb{C}$

3. the unit disk in the complex plane, $\mathbb{D}$

It has been shown that when Σ is a compact Riemann surface of genus 2 or greater, its universal covering is biholomorphic to the unit disk $\mathbb{D}$. The fundamental domain of Σ would then tessellate $\mathbb{D}$. Hence, using the Poincaré disk model, one only needs to find holomorphic maps from the fundamental domain with suitable ramification points to create vortex solutions. Maldonado and Manton's method does so by making heavy use of the Riemann Mapping Theorem, Carathéodory's Theorem, and the Schwarz Reflection Principle [7]. The statements of these theorems are stated below without proof [3] [4].

Theorem 2.0.2 (Riemann Mapping Theorem)**.** *If* U *is a simply connected subset of the complex plane* $\mathbb{C}$ *which is not the entire complex plane, then there exists a biholomorphic mapping* f *(called the Riemann mapping) from the open unit disk* $\mathbb{D}$ *to* U*.*

Theorem 2.0.3 (Carathéodory's Theorem)**.** *If* f *maps the open disk* $\mathbb{D}$ *conformally onto a bounded domain* $U \subset \mathbb{C}$*, then* f *has a continuous one-to-one extension to the closed disk* $\overline{\mathbb{D}}$ *if and only if* ∂U *is a Jordan curve.*

It should be noted here that the unit disk $\mathbb{D}$ and its closure $\overline{\mathbb{D}}$ in the statement of the above theorems may be replaced by the extended upper half-plane $\mathbb{H}$ and its closure $\overline{\mathbb{H}}$

respectively. This is because f may be precomposed with the Möbius transformation

$$g(z) = \frac{z-i}{z+i} \tag{28}$$

which maps $\mathbb{H}$ onto $\mathbb{D}$ conformally.

Theorem 2.0.4 (Schwarz Reflection Principle)**.** *Suppose f is holomorphic on the upper half-plane $\mathbb{H}$, has a continuous extension to an interval $I \subset \mathbb{R}$ with $f(I) \subset \mathbb{R}$. Extend f to the lower half-plane $\mathbb{L} = \{z : \overline{z} \in \mathbb{H}\}$ by setting*

$$f(z) = \overline{f(\overline{z})}, \qquad z \in \mathbb{L} \tag{29}$$

then the extended function is holomorphic on $\mathbb{H} \cup I \cup \mathbb{L}$.

The desired holomorphic mapping is constructed by applying the Riemann Mapping Theorem and Carathéodory's Theorem to hyperbolic polygons as they satisfy the conditions of both theorems. Without loss of generality, the Schwarz Reflection Principle may also be applied about any edge of the polygon. This is because one can map the edge onto the real axis using a Möbius transformation, reflect it, and then apply the inverse Möbius transformation to map the edge back onto itself. The aim of this procedure is to construct a holomorphic mapping from the unit disk $\mathbb{D}$ to the upper half-plane $\mathbb{H}$ given a tessellation of the unit disk and a conformal mapping $f : \mathbb{H} \to P$ where P is a hyperbolic polygon.

2.5.1 Hyperbolic Polygons

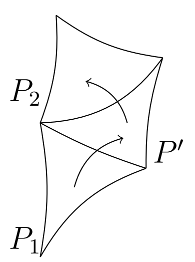
To get the holomorphic map h, hyperbolic polygons in the Poincaré disk model are considered. Geodesics in this model are either lines passing through the origin, or circles intersecting the boundary of the unit disk $\partial\mathbb{D}$ at right angles. Hyperbolic n-gons are defined to be open subsets of $\mathbb{D}$ bounded by n geodesics. Before h can be constructed, some preliminary results and technical lemmas will be needed. These will be proved in Appendix A.

Proposition 2.0.5. *Let P be a hyperbolic polygon. Let $r(z)$ be the map reflecting P over one of its edges E and let $g(z)$ be a Möbius transformation mapping E onto the real axis. Then $r = g^{-1} \circ \overline{g}$ and $r(P)$ is also a hyperbolic polygon.*

Lemma 2.0.6. *The composition of an even number of reflections may be expressed as a Möbius transformation.*

Proposition 2.0.7. *Let P_1 be a hyperbolic polygon and let P_2 be the image of P_1 under two successive Schwarz reflections equivalent to a Möbius transformation g. Let $f_1 : \mathbb{H} \to P_1$ be a Riemann mapping onto P_1 and $f_2 : \mathbb{H} \to P_2$ be a Riemann mapping onto P_2 given by $f_2 = g \circ f_1$. Denote their respective extensions (via the Schwarz Reflection Principle) onto the intermediate polygon P' by F_1 and F_2 respectively. Then F_1^{-1} and F_2^{-1} agree on P' (see figure 1 below).*

Figure 1: Successive Schwarz Reflections



We prove one final result regarding biholomorphic maps and the Schwarz Reflection

Principle.

Proposition 2.0.8. *Let f be a Riemann mapping from the upper half-plane $\mathbb{H}$ onto an open set $U \subset \mathbb{C}$. Furthermore, suppose that an interval $I \in \mathbb{R}$ is a segment of the boundary of U (i.e. $I \subset \partial U \cap \mathbb{R}$). Then the Schwarz Reflection of f over I is also a biholomorphic map.*

Proof. By Carathéodory's Theorem, f may be extended onto its boundary continuously. In particular, an interval $I \in \mathbb{R}$ is a subset of ∂U the Schwarz Reflection Principle may be applied to f. By the statement of the Reflection Principle, its reflection (also denoted f) is holomorphic on $\mathbb{H} \cup I \cup \mathbb{L}$. Since f is biholomorphic, for any $z \in \mathbb{H}$, we have that $f'(z) \neq 0$.

For $z \in \mathbb{L}$, we see that

$$\begin{aligned} f'(z) &= \lim_{|h| \to 0} \frac{f(z+h) - f(z)}{h} \\ &= \lim_{|h| \to 0} \frac{\overline{f(\overline{z+h})} - \overline{f(\overline{z})}}{h} \\ &= \lim_{|\overline{h}| \to 0} \overline{\left(\frac{f(\overline{z}+\overline{h}) - f(\overline{z})}{\overline{h}} \right)} \\ &= \overline{f'(\overline{z})} \end{aligned} \tag{30}$$

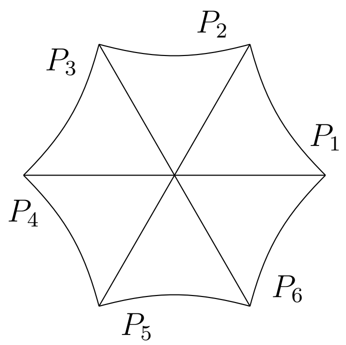
Hence $f'(z) \neq 0$ for each $z \in \mathbb{H} \cup \mathbb{L}$ and thus f is conformal and biholomorphic on $\mathbb{H} \cup \mathbb{L}$.

For $z \in I$, since f is holomorphic and one-to-one on I, it is invertible and hence biholomorphic on I. Thus the Schwarz Reflection of f over I is biholomorphic. □

2.5.2 Construction of a Vortex Solutions

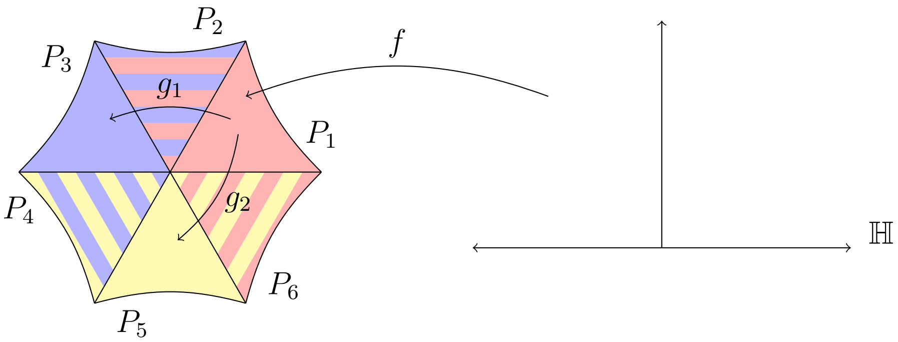
Results from the previous section allow for the construction of the desired map h. Suppose T is a tessellation of the Poincaré disk using hyperbolic n-gons. We may divide an n-gon P into congruent polygonal sections $P_1, ..., P_k$ using geodesics from its center to its boundary (see figure 2). By the Riemann Mapping Theorem, there exists a biholomorphic mapping f_1 from the upper half-plane $\mathbb{H}$ onto P_1. By Carathéodory's theorem f_1 extends continuously onto its boundary. Using the Schwarz Reflection Principle, f_1 may be extended further onto other sections sharing an edge with P_1. For example, in the diagram below, the map $f_1 : \mathbb{H} \to P_1$ has, by Schwarz reflection, extensions onto P_2 and P_6. It should also be noted that though it is not depicted in the diagram, f_1 would also have an extension onto another triangular region in another n-gon by reflection over its remaining edge.

Figure 2: Dividing an n-gon into congruent polygonal sections



Multiple maps from the complex plane $\mathbb{C}$ onto different areas of the Poincaré disk are attained by composing f_1 with various Möbius transformations. The ones of particular interest are the ones composed with Möbius transformations equivalent to an even number of reflections (see figure 3). For example, in the diagram below, $g_1 \circ f$ and f overlap on P_2, $g_1 \circ f$ and $g_2 \circ f$ overlap on P_4, and $g_2 \circ f$ and f overlap on P_6.

Figure 3: Overlapping Conformal Maps from Möbius Transformations



It is clear that each of these maps are multi-valued since Schwarz reflections are applied to each edge of the polygonal section, however the maps are conformal when restricted to a single section. Hence the inverse maps are single-valued holomorphic functions. Proposition 2.0.7 shows that the procedure of composing f with Möbius transformations in general ensures that the inverse maps agree on their common domains. By the Indentity Theorem, they may be combined into a single holomorphic function F.

So far, this method has left out the vertices of each of the polygonal sections. However, since F was constructed via Schwarz reflection around each vertex, it is clear that F is continuously extendable onto the vertices of each polygonal section. The Riemann Extension Theorem asserts that F can also be extended holomorphically extended onto the vertices of the tessellation T. Hence F is a holomorphic map from $\mathbb{D}$ onto $\mathbb{H}$ satisfying compatibility requirements with Schwarz reflection and a tessellating group action on $\mathbb{D}$.

The same technique may be applied to another hyperbolic tessellation T' of $\mathbb{D}$ using m-gons. From this another holomorphic map $G : \mathbb{D} \to \mathbb{H}$ satisfying similar compatibility requirements is obtained. By selecting a base polygonal section Q_1 of the new tessellating m-gon Q, it is possible to define a new map h, which maps P_1 onto Q_1 by requiring that

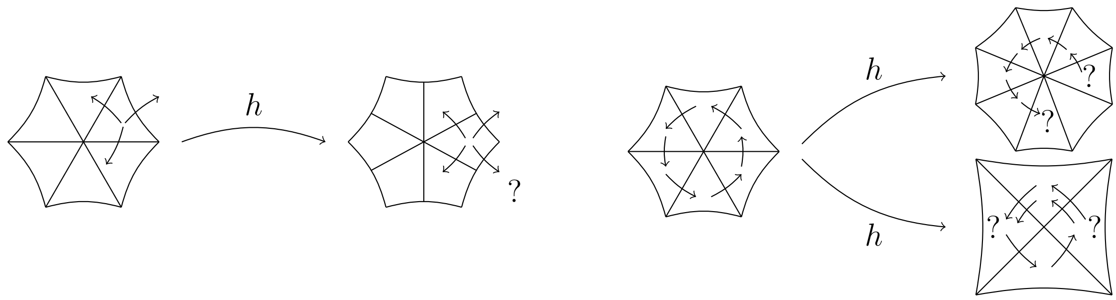
$$h|_{P_1} = (G|_{Q_1})^{-1} \circ F|_{P_1} \tag{31}$$

where $(G|_{Q_1})^{-1}$ denotes the inverse of G when restricted to Q_1. Even after additionally requiring that h preserve the reflection action on each of the tessellating polygons (i.e., $h \circ r_1|_{P_i} = r_2|_{Q_j} \circ h$ where r_1 and r_2 are reflections over corresponding edges of corresponding polygonal sections P_i and Q_j of P and Q respectively), it can be seen that with certain restrictions, h is still well-defined. From the properties of both F and G, it can be seen that h is holomorphic and conformal everywhere on $\mathbb{D}$ except at the vertices of the tessellation T.

As mentioned previously, there are some conditions that need to be met in order to assure that h will be well-defined. Firstly, the polygonal sections must both have the same number of edges and vertices, else one of them will have additional possible reflections. This means that Schwarz reflection cannot be compatible with the map. Secondly, m must divide n, if not Schwarz reflecting P_1 around the n-gon such that it returns to its starting position would cause h to be multi-valued (see figure 4).

Figure 4: Restrictions of the Method

(a) Polygonal sections must be the same (b) m divides n



The method outlined above merely provides the existence of symmetric vortex solutions on certain compact Riemann surfaces. Maldonado and Manton managed to calculate the explicit form of some of these solutions by restricting their attention to hyperbolic triangles [7], where the Riemann mapping is known and can be expressed in terms of hypergeometric functions [4] [6]. Additionally, by forcing a doubling of angles at the center and vertices, they were able to ensure that the map h and the Higgs field ϕ had zeros there as well.

3.0 Extensions and Generalizations

Manton altered the Bogomolny vortex equations by allowing some of the coefficients to take arbitrary values [8]. The equations he considered were of the form

$$\frac{-2i}{\Omega} F_{z\bar{z}} = -C + C_2|\phi|^2 \tag{32}$$

$$D_{\bar{z}}\phi = 0 \tag{33}$$

Though this now results in a two-parameter family of solutions, by rescaling the connection A and the Higgs field ϕ, only five types of vortices remain. They are listed below:

Standard (Taubes)	$C = -1, C_2 = -1$
Bradlow	$C = -1, C_2 = 0$
Ambjørn-Olesen	$C = -1, C_2 = 1$
Jackiw-Pi	$C = 0, C_2 = 1$
Popov	$C = 1, C_2 = 1$

Table 1: The Five Types of Vortices

By also considering a modified Yang-Mills-Higgs energy functional

$$E_{C,C_2}(A,\phi) = \int_\Sigma \left[(*F)^2 - \frac{C}{\Omega}|D\phi|^2 + (-C_0 - C|\phi|^2)^2 \right] \frac{i}{2}\Omega dz \wedge d\bar{z} \tag{34}$$

it can be shown that the solutions to the new vortex equations are critical points of the functional, though not necessarily minima. The latter can be seen by applying the same method used in Section 2.1.

Using Euclidean tessellations and Schwarz-Christoffel mappings, it may be possible to create Bradlow vortices on specific Riemann surfaces using the method outlined above. Additionally, the same might be possible for spherical tessellations and Ambjørn-Olesen vortices. Both these cases are more restrictive since there are only finitely many Euclidean and spherical tessellations. Another possible consideration is whether it is possible to relax the symmetry constraint imposed on the solutions by supposing other tilings of the fundamental domain.

4.0 Conclusion

This paper has outlined a method of Maldonado and Manton [7] used to create vortex solutions on compact hyperbolic Riemann surfaces. This method shows the existence of symmetric vortex solutions on hyperbolic Riemann surfaces meeting certain requirements. It, however, does not provide an explicit form for the vortex unless the Riemann mapping of the polygon being used is known explicitly as well.

Though this method was first applied onto hyperbolic surfaces, it may be possible to alter it slightly and use it on other types of surfaces to create new vortex solutions. Additionally, the symmetry imposed onto the solutions via the construction might be possible to relax by altering Maldonado and Manton's method.

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Appendix A Proofs of Propositions and Lemmas

Proof of Proposition 2.0.5

Proof. Since P is a hyperbolic polygon, its edges are either segments of lines which pass through the origin or circles perpendicular to the unit disk. Consider the two cases:

Case 1: E lies on a line L: Since L passes through the origin, L is specified by an angle θ, so $L = \{z \in \mathbb{C} : z = te^{i\theta}, t \in \mathbb{R}\}$. Let a, b, c be distinct points on L, so $a = a_0 e^{i\theta}, b = b_0 e^{i\theta}, c = c_0 e^{i\theta}$ with $a_0, b_0, c_0 \in \mathbb{R}$. Consider the cross-ratio

$$\begin{aligned} g(z) = [z; a, b, c] &= \frac{(z-a)(b-c)}{(z-c)(b-a)} \\ &= \frac{(z-a_0e^{i\theta})(b_0e^{i\theta}-c_0e^{i\theta})}{(z-c_0e^{i\theta})(b_0e^{i\theta}-a_0e^{i\theta})} \\ &= \frac{(z-a_0e^{i\theta})(b_0-c_0)}{(z-c_0e^{i\theta})(b_0-a_0)} \end{aligned} \tag{35}$$

If $p = p_0 e^{i\theta} \in L$ then,

$$\begin{aligned} g(p) &= \frac{(p_0e^{i\theta}-a_0e^{i\theta})(b_0-c_0)}{(p_0e^{i\theta}-c_0e^{i\theta})(b_0-a_0)} \\ &= \frac{(p_0-a_0)(b_0-c_0)}{(p_0-c_0)(b_0-a_0)} \in \mathbb{R} \end{aligned} \tag{36}$$

So g maps L onto $\mathbb{R}$ and the only Möbius transformations which do so are of the above form.

Reflection of a point z over L is given by

$$r(z) = e^{2i\theta}\overline{z} \tag{37}$$

Evaluating gives

$$
\begin{aligned}
g \circ r(z) &= g(e^{2i\theta}\overline{z}) \\
&= \frac{(\overline{z}e^{2i\theta} - a_0e^{i\theta})(b_0 - c_0)}{(\overline{z}e^{2i\theta} - c_0e^{i\theta})(b_0 - a_0)} \\
&= \frac{(\overline{z} - a_0e^{-i\theta})(b_0 - c_0)}{(\overline{z} - c_0e^{-i\theta})(b_0 - a_0)} \\
&= \frac{\overline{(z - a_0e^{i\theta})}\,\overline{(b_0 - c_0)}}{\overline{(z - c_0e^{i\theta})}\,\overline{(b_0 - a_0)}} \\
&= \overline{g(z)}
\end{aligned}
\tag{38}
$$

Case 2: E lies on a line C: Suppose C has center z_0 and radius r. Again, let a, b, c be distinct points on C, so $a = z_0 + re^{i\theta_a}, b = z_0 + re^{i\theta_b}, c = z_0 + re^{i\theta_c}$. Once more, consider the cross-ratio

$$
\begin{aligned}
g(z) = [z; a, b, c] &= \frac{(z-a)(b-c)}{(z-c)(b-a)} \\
&= \frac{(z - (z_0 + re^{i\theta_a}))((z_0 + re^{i\theta_b}) - (z_0 + re^{i\theta_c}))}{(z - (z_0 + re^{i\theta_c}))((z_0 + re^{i\theta_b}) - (z_0 + re^{i\theta_a}))} \\
&= \frac{(z - (z_0 + re^{i\theta_a}))(e^{i\theta_b} - e^{i\theta_c})}{(z - (z_0 + re^{i\theta_c}))(e^{i\theta_b} - e^{i\theta_a})}
\end{aligned}
\tag{39}
$$

If $p = z_0 + re^{i\theta_p} \in C$, then

$$
\begin{aligned}
g(p) &= \frac{((z_0 + re^{i\theta_p}) - (z_0 + re^{i\theta_a}))(e^{i\theta_b} - e^{i\theta_c})}{((z_0 + re^{i\theta_p}) - (z_0 + re^{i\theta_c}))(e^{i\theta_b} - e^{i\theta_a})} \\
&= \frac{(e^{i\theta_p} - e^{i\theta_a})(e^{i\theta_b} - e^{i\theta_c})}{(e^{i\theta_p} - e^{i\theta_c})(e^{i\theta_b} - e^{i\theta_a})}
\end{aligned}
\tag{40}
$$

It can be seen that

$$\overline{g(p)} = \frac{(e^{-i\theta_p} - e^{-i\theta_a})(e^{-i\theta_b} - e^{-i\theta_c})}{(e^{-i\theta_p} - e^{-i\theta_c})(e^{-i\theta_b} - e^{-i\theta_a})} \tag{41}$$

$$= \frac{(\frac{e^{i\theta_a} - e^{i\theta_p}}{e^{i\theta_p} e^{i\theta_a}})(\frac{e^{i\theta_c} - e^{i\theta_b}}{e^{i\theta_b} e^{i\theta_c}})}{(\frac{e^{i\theta_c} - e^{i\theta_p}}{e^{i\theta_p} e^{i\theta_c}})(\frac{e^{i\theta_a} - e^{i\theta_b}}{e^{i\theta_b} e^{i\theta_a}})} \tag{42}$$

$$= \frac{(e^{i\theta_a} - e^{i\theta_p})(e^{i\theta_c} - e^{i\theta_b})}{(e^{i\theta_c} - e^{i\theta_p})(e^{i\theta_a} - e^{i\theta_b})} \tag{43}$$

$$= g(p) \tag{44}$$

Hence $g(p) \in \mathbb{R}$ and so g maps C (and hence E) onto $\mathbb{R}$. Once again, the only Möbius transformations mapping C to $\mathbb{R}$ are of the above form.

The reflection of a point z over C is given by

$$r(z) = \frac{r^2}{\overline{z - z_0}} + z_0 \tag{45}$$

Evaluating gives

$$
\begin{aligned}
g \circ r(z) &= g\left(\frac{r^2}{\overline{z} - \overline{z_0}} + z_0\right) &(46)\\
&= \frac{\left(\frac{r^2}{\overline{z}-\overline{z_0}} + z_0 - (z_0 + re^{i\theta_a})(e^{i\theta_b} - e^{i\theta_c}\right)}{\left(\frac{r^2}{\overline{z}-\overline{z_0}} + z_0 - (z_0 + re^{i\theta_c})(e^{i\theta_b} - e^{i\theta_a}\right)} &(47)\\
&= \frac{\left(\frac{r}{\overline{z}-\overline{z_0}} - e^{i\theta_a}\right)\left(e^{i\theta_b} - e^{i\theta_c}\right)}{\left(\frac{r}{\overline{z}-\overline{z_0}} - e^{i\theta_c}\right)\left(e^{i\theta_b} - e^{i\theta_a}\right)} &(48)\\
&= \frac{\overline{\left(\frac{r}{z-z_0} - e^{-i\theta_a}\right)}\;\overline{\left(e^{-i\theta_b} - e^{-i\theta_c}\right)}}{\overline{\left(\frac{r}{z-z_0} - e^{-i\theta_c}\right)}\;\overline{\left(e^{-i\theta_b} - e^{-i\theta_a}\right)}} &(49)\\
&= \frac{\overline{\left(\frac{re^{i\theta_a}-(z-z_0)}{(z-z_0)e^{i\theta_a}}\right)}\;\overline{\left(\frac{e^{i\theta_c}-e^{i\theta_b}}{e^{i\theta_b}e^{i\theta_c}}\right)}}{\overline{\left(\frac{re^{i\theta_c}-(z-z_0)}{(z-z_0)e^{i\theta_c}}\right)}\;\overline{\left(\frac{e^{i\theta_a}-e^{i\theta_b}}{e^{i\theta_b}e^{i\theta_a}}\right)}} &(50)\\
&= \frac{\overline{\left(z - (z_0 + re^{i\theta_a})\right)}\;\overline{\left(e^{i\theta_b} - e^{i\theta_c}\right)}}{\overline{\left(z - (z_0 + re^{i\theta_c})\right)}\;\overline{\left(e^{i\theta_b} - e^{i\theta_a}\right)}} &(51)\\
&= \overline{g(z)} &(52)
\end{aligned}
$$

Again, $r = g^{-1} \circ \overline{g}$.

To show that $r(P)$ is also a hyperbolic polygon, use the fact that Möbius transformations and reflections map lines and circles onto other lines and circles to get that $r(P)$ is bounded by lines and circles in the complex plane. In either case, by geometrical arguments, the boundary of the unit disk $\partial\mathbb{D}$ must be mapped onto itself by the reflection. Since the edge is fixed by the reflection and lies inside the unit disk $\mathbb{D}$, by continuity, $\mathbb{D}$ must also be mapped onto itself by r. Thus $r(P) \subset \mathbb{D}$. Furthermore, since Möbius transformations and reflections are angle-preserving, the images of the other edges must lie on lines or circles which intersect $\partial\mathbb{D}$ at right angles and are hence geodesics in the Poincaré disk model. Hence $r(P)$ is a hyperbolic polygon. $\square$

Proof of Lemma 2.0.6

Proof. By Proposition 2.0.5 the reflections r_1 and r_2 may be written as $r_1 = f^{-1} \circ \overline{f}, r_2 = g^{-1} \circ \overline{g}$ where f and g are Möbius transformations. Thus,

$$r_2 \circ r_1(z) = g^{-1}(\overline{g(f^{-1}(\overline{f(z)}))}) \tag{53}$$

Without loss of generality, suppose that f and g have the forms

$$f(z) = \frac{az + b}{cz + d} \tag{54}$$

$$g(z) = \frac{\alpha z + \beta}{\gamma z + \delta} \tag{55}$$

where $a, b, c, d, \alpha, \beta, \gamma, \delta \in \mathbb{C}$. Using this, it can be shown that

$$f^{-1}(\overline{f(z)} = \frac{(\overline{a}d - b\overline{c})\overline{z} + (\overline{b}d - b\overline{d})}{(a\overline{c} - \overline{a}c)\overline{z} + (a\overline{d} - \overline{b}c)} \tag{56}$$

$$\overline{g(f^{-1}(\overline{f(z)}))} = \frac{\overline{[\alpha(\overline{a}d - b\overline{c}) + \beta(a\overline{c} - \overline{a}c)]}z + \overline{[\alpha(\overline{b}d - b\overline{d}) + \beta(a\overline{d} - \overline{b}c)]}}{\overline{[\gamma(\overline{a}d - b\overline{c}) + \delta(a\overline{c} - \overline{a}c)]}z + \overline{[\gamma(\overline{b}d - b\overline{d}) + \delta(a\overline{d} - \overline{b}c)]}} \tag{57}$$

Hence $\overline{g \circ^{-1} \circ \overline{f}}$ is a Möbius transformation. Since the compositions of Möbius transformations are also Möbius transformations, $r_2 \circ r_1 = g^{-1} \circ \overline{g \circ^{-1} \circ \overline{f}}$ is a Möbius transformation. □

Proof of Proposition 2.0.7

Proof. Denote the two reflections r_1 and r_2, so $g = r_2 \circ r_1$. Let h and k be Möbius transformations mapping the reflecting edges of P_1 and P_2 respectively onto the real line.

Hence the Schwarz reflections F_1 and F_2 are given by

$$F_1 = h^{-1} \circ \overline{h(f_1)} \tag{58}$$

$$F_2 = k^{-1} \circ \overline{k(f_2)} = k^{-1} \circ \overline{k(g(f))} \tag{59}$$

Proposition 2.0.5 says that $r_1 = h^{-1} \circ \overline{h}$ and that $r_2^{-1} = k^{-1} \circ \overline{k}$. Hence

$$F_1 = r_1 \circ f_1 \tag{60}$$

$$F_2 = r_2^{-1} \circ g \circ f_1 = r_1 \circ f_1 \tag{61}$$

Thus F_1^{-1} and F_2^{-1} agree on P'.

$\square$